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Unified Treatment of Lifting Atmospheric Entry

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This paper presents a unified treatment of the effect of lift on peak acceleration during atmospheric entry. Earlier studies were restricted to different regimes because of approximations invoked to solve the same transcendental equation. This paper shows the connection between the earlier studies by employing a general expression for the peak acceleration and obtains solutions to the transcendental equation without invoking the earlier approximations. Results are presented and compared with earlier studies where appropriate.

Nomenclature

A = vehicle frontal area
C_D = drag coefficient
D = drag
g = gravitational acceleration
G = acceleration measured in units of Earth gravity
L = lift
m = vehicle mass
R = distance from planet's center to vehicle
V = velocity
y = altitude above planet's surface
β = inverse scale height = 1/23,000 ft^{-1} = 1/7.01 km^{-1} (Earth)
θ = angle between local horizontal and V
ρ = density

Subscript

E = entry conditions

Superscript

$(\ )^*$ = conditions at peak acceleration

Introduction

ANALYTIC studies dealing with the effect of lift on peak acceleration during atmospheric entry have been presented by Lees et al.[1] and Arthur and Karrenberg.[2] The describing equations in both studies are the same, but the results of each are restricted to different regimes. Reference 1 is restricted to small entry angles and Ref. 2 to small values of L/D. The purpose of the present paper is to provide a unified treatment. It is pointed out herein that the restrictions imposed in the cited references arise in the solution of the same transcendental equation. This transcendental equation does not appear explicitly in the work of Chapman[3] since his results are obtained by numerical integration of the equations of motion. The present work deals with the effect of lift on peak acceleration and is restricted to entry velocities in the neighborhood of the circular satellite velocity.

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Analysis

Using the notation of Ref. 1, the describing equations of the motion of a body in the atmosphere of a planet are

$$m\dot{V} = -D + mg\sin\theta \tag{1}$$

$$mV\dot{\theta} = -L + mg\cos\theta - (mV^2\cos\theta)/R \tag{2}$$

As in the Allen and Eggers study,[4] the assumptions are: weight neglected, flat earth ($R \to \infty$), and constant g; thus the equations reduce to

$$\dot{V} = -D/m \tag{3}$$

$$V\dot{\theta} = -L/m \tag{4}$$

These are the common equations used in Refs. 1 and 2 for a constant lift-drag ratio.

From the exponential approximation for the atmospheric density, $\rho = \rho_0 \exp(-\beta y)$, it follows that $\dot{\rho} = -\beta\dot{y}$; this, coupled with the relation $\dot{y} = -V\sin\theta$ and $D = \frac{1}{2}\rho C_D A V^2$, yields

$$\mathrm{d}V^2/\mathrm{d}\rho = -C_D A V^2/\beta m \sin\theta \tag{5}$$

$$\mathrm{d}\cos\theta/\mathrm{d}\rho = \tfrac{1}{2}(L/D)(C_D A/\beta m) \tag{6}$$

Dividing Eq. (5) by Eq. (6) leads to the integral

$$V^2 = V_E^2 \exp\frac{\theta - \theta_E}{\frac{1}{2}L/D} \tag{7}$$

The integral of Eq. (6) is (for $\rho_E \doteq 0$)

$$\rho = \frac{\beta m}{C_D A}\,\frac{\cos\theta - \cos\theta_E}{\frac{1}{2}L/D} \tag{8}$$

When these relations are substituted into the expression for the acceleration $\dot{V} = -\frac{1}{2}\rho(C_D A V^2/m)$, the acceleration is expressed as a function of θ alone. Differentiation and equating to zero the formula for the acceleration leads to the following expression:

$$\tfrac{1}{2}L/D\sin\theta = \cos\theta - \cos\theta_E \tag{9}$$

which holds at peak acceleration. The sudies in Refs. 1 and 2 differed in the assumptions invoked to solve Eq. (9). Reference 1 considered the case $\theta_E \ll 1$ and Ref. 2 the case $L/D \ll 1$ or, equivalently, in view of Eq. (9), $|\theta - \theta_E| \ll 1$. In addition, Allen and Eggers work[4] for ballistic entry

$(L/D=0)$ satisfied Eq. (9) by assuming $\theta=\theta_E$. In all cases, from Eqs. (8) and (9), it follows that a peak acceleration

$$\rho^* = \frac{\sin\theta^*}{C_D A/\beta m} \tag{10}$$

where θ denotes the solution of Eq. (9) and the asterisk indicates values at peak acceleration. Also

$$-\dot{V}^* = \tfrac{1}{2}\beta \sin\theta^* V_E^2 \exp\frac{\theta^*-\theta_E}{\tfrac{1}{2}L/D} \tag{11}$$

where $\dot{V}=-(D/m)$ is the acceleration along the flight path and $V\dot{\theta}=-(L/m)$ is the acceleration normal to the flight path. The resultant of these (measured units of Earth g) is given by

$$G=D/gm\sqrt{1+(L/D)^2} = -(\dot{V}/g)\sqrt{1+(L/D)^2}$$

or at peak acceleration

$$G^* = \frac{1}{2}\frac{\beta}{g}\sin\theta^* V_E^2\left(\exp\frac{\theta^*-\theta_E}{\tfrac{1}{2}L/D}\right)\sqrt{1+\left(\frac{L}{D}\right)^2} \tag{12}$$

Consistent with the approximations used to derive Eqs. (3) and (4), this is the general expression for peak acceleration. The various expressions used in earlier studies can be derived from this relation. First consider the case of Allen and Eggers,[4] $L/D=0$, $\theta=\theta_E$; $(\theta^*-\theta_E)/(\tfrac{1}{2}L/D)$ is indeterminate. However, at peak acceleration, Eq. (9) gives L/D as a function of θ^*. This permits evaluation of the indeterminate form by L'Hospital's rule since

$$\frac{d}{d\theta^*}\left(\frac{1}{2}\frac{L}{D}\right) = \frac{d}{d\theta^*}\frac{\cos\theta^*-\cos\theta_E}{\sin\theta^*}$$

At $\theta^*=\theta_E$ the derivative equals -1, which gives

$$G^*_{(L/D=0)} = \tfrac{1}{2}(\beta/ge)\sin\theta_E V_E^2 \tag{13}$$

Lees et al.[1] restricted the solution of Eq. (9) to the region $\theta_E \ll 1$. This permits the use of the approximation $\sin\theta \doteq \theta$ and $\cos\theta \doteq 1-(\theta^2/2)$. With these approximations, Eq. (9) is written

$$\theta^2 + (L/D)\theta = \theta_E^2$$

with an approximate solution $\theta^* = \theta_E^2/(L/D)$. When substituted into Eq. (12), this approximate solution yields

$$G^* = \frac{1}{2}\frac{\beta}{g}\frac{\theta_E^2}{L/D}V_E^2\left(\exp-\frac{\theta_E}{\tfrac{1}{2}L/D}\right)\sqrt{1+\left(\frac{L}{D}\right)^2} \tag{14}$$

Note that the solution considered by Lees et al. rules out the case of vanishing L/D. As shown earlier, this is the Allen-Eggers solution.

Arthur and Karrenberg[2] restrict the solution of Eq. (9) to the region $L/D \ll 1$ or, equivalently, $|\theta-\theta_E| \ll 1$. The terms in Eq. (9) are expanded in powers of $\theta-\theta_E$ as follows:

$$\sin\theta = \sin\theta_E[1+(\theta-\theta_E)\operatorname{ctn}\theta_E] \tag{15}$$

$$\cos\theta-\cos\theta_E = -(\theta-\theta_E)\sin\theta_E\left(1+\frac{\theta-\theta_E}{2}\operatorname{ctn}\theta_E\right)$$

Substituting into Eq. (9) yields

$$\frac{1}{2}\frac{L}{D}[1+(\theta-\theta_E)\operatorname{ctn}\theta_E] = -(\theta-\theta_E)\left(1+\frac{\theta-\theta_E}{2}\operatorname{ctn}\theta_E\right)$$

An approximate solution is

$$\theta^*-\theta_E = -\tfrac{1}{2}(L/D)(1-\tfrac{1}{4}(L/D)\operatorname{ctn}\theta_E) \tag{16}$$

The above equation can be seen to be equivalent to Eq. (12) in Ref. 2 when $(\theta-\theta_E)$ is eliminated between Eqs. (7) and (16) of the present paper. Substituting Eqs. (15) and (16) into (12) after expanding the exponential and simplifying by neglecting higher powers of L/D yields

$$G^* = \frac{\beta\sin\theta_E}{2eg}V_E^2\left(1-\frac{1}{4}\frac{L}{D}\operatorname{ctn}\theta_E\right)\sqrt{1+\left(\frac{L}{D}\right)^2} \tag{17}$$

It has been demonstrated that the general relation Eq. (12) yields the previous specialized cases, and that the restrictions in these special cases are imposed in solving Eq. (9). Finally, consider the expression given by Chapman[3] for peak acceleration for shallow entry angles. Equation (12) can be written at peak accleration as

$$G^* = \frac{1}{2}\frac{\beta}{g}\frac{C_D A}{\beta m}\rho^* V^{*2}\sqrt{1+\left(\frac{L}{D}\right)^2}$$

where Eqs. (7) and (10) have been used to eliminate expressions containing θ^*.

Rearranging and introducing Chapman's notation (R = distance from planet's center to vehicle, $\sqrt{gR}$ = circular satellite velocity), we have

$$G^* = \sqrt{\beta R}\frac{V^*}{\sqrt{gR}}\left[\sqrt{\frac{R}{\beta}}\rho^*\frac{C_D A}{2m}\frac{V^*}{\sqrt{gR}}\right]\sqrt{1+\left(\frac{L}{D}\right)^2} \tag{18}$$

which is Chapman's result for peak acceleration [Eq. (51), Ref. 3]. The expression in brackets in the above equation is Chapman's Z function evaluated at peak acceleration for shallow entry angles ($\cos\theta \doteq 1$). At this point, the distinction between Chapman's work and the present treatment should be made clear. To evaluate Eq. (18), Chapman made use of the solutions of the equations of motion obtained by numerical integration. In the present paper, the evaluations are made by obtaining the solution to Eq. (9). In addition, the solutions obtained herein are not restricted to shallow entry angles.

The approximate graphic solution of Eq. (9) is interesting since it can show the nature of the approximations invoked by the previous investigations to solve it. To illustrate the graphic solution of Eq. (9), write it as follows:

$$\tfrac{1}{2}(L/D)\sin\theta + \cos\theta_E = \cos\theta \tag{9'}$$

By plotting the left- and right-hand sides of the above equation, the point of intersetion of the two curves will correspond to θ^*, the solution (Fig. 1). For a given entry angle θ_E, the analytic solutions of Ref. 1 are restricted to values that are close to zero and those of Ref. 2, near θ_E. In terms of the restrictions on L/D, solutions of Ref. 1 cannot permit vanishing L/D and solutions of Ref. 2 are restricted to small values of L/D. It is important to emphasize that these restrictions are lifted if Eq. (9′) is solved without invoking any approximations.

Equation (9′) can easily be solved numerically by Newton's method since the appropriate derivative is easily obtained. In fact, if θ is an approximate solution of Eq. (9′), $\theta+\Delta\theta$ is a better approximation where

$$\Delta\theta = -\frac{\tfrac{1}{2}L/D\,\sin\theta - \cos\theta + \cos\theta_E}{\tfrac{1}{2}L/D\,\cos\theta + \sin\theta}$$

For given values of θ_E and L/D, the solution of Eq. (9′) was carried out with a small programmable calculator. An analytic solution to Eq. (9′) can be obtained by transposing the term $\cos\theta_E$ and squaring both sides. This leads to a

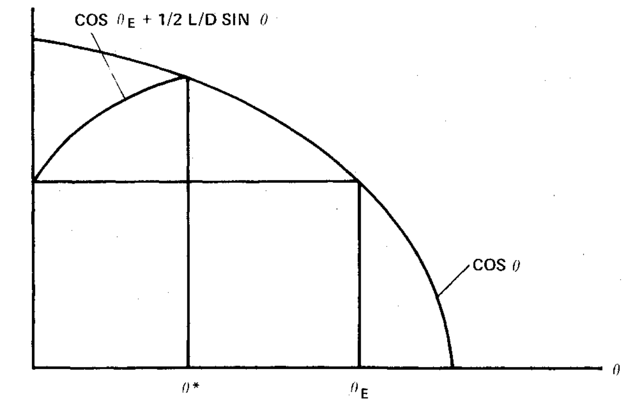

Fig. 1 Solution of Eq. (9′).

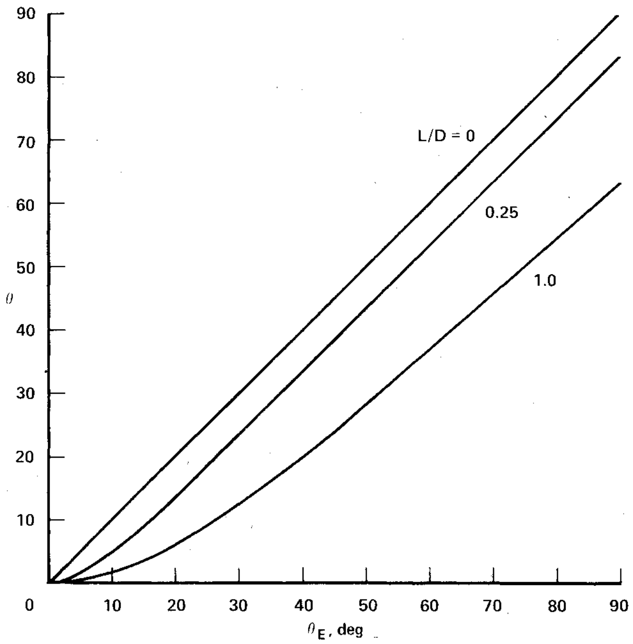

Fig. 2 Solution of ½ L/D sinθ = cosθ − cosθ_E.

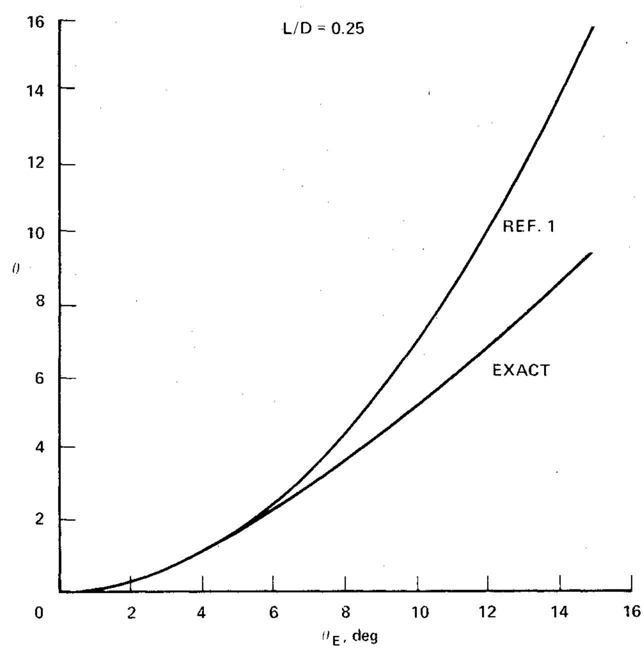

Fig. 3 Solution of ½ L/D sinθ = cosθ − cosθ_E for L/D = 0.25.

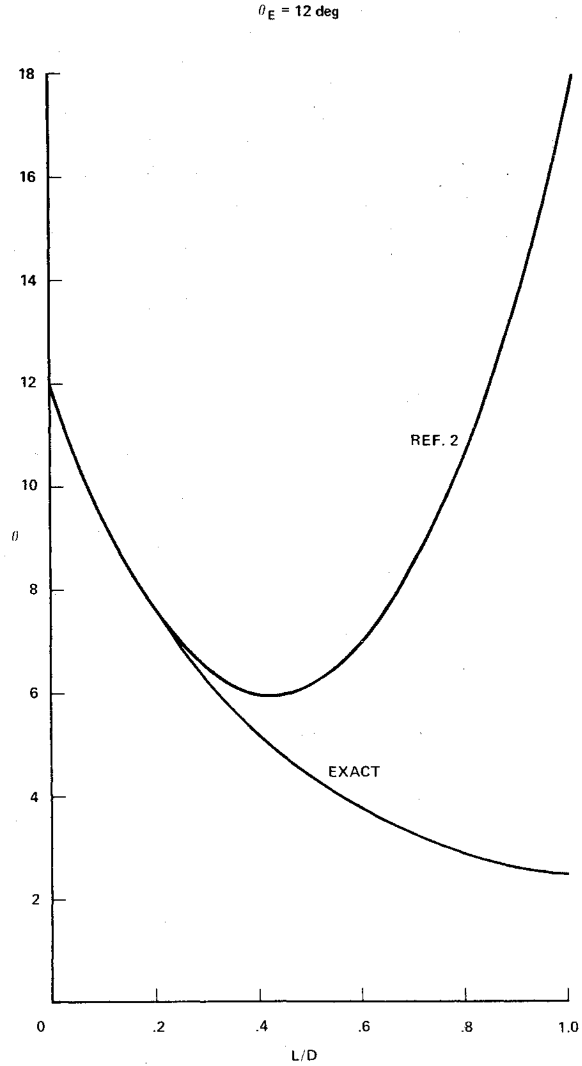

Fig. 4 Solution of ½ L/D sinθ = cosθ − cosθ_E for θ_E = 12 deg.

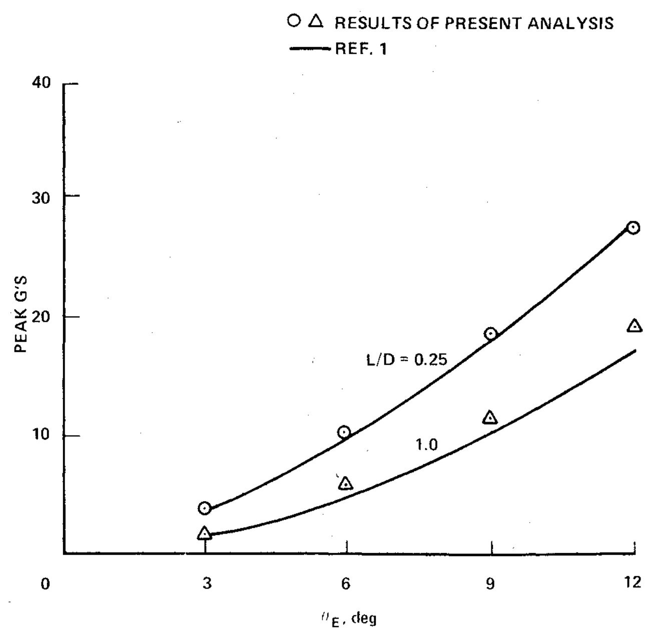

Fig. 5 Peak acceleration, V_E = 26,000 fps (7.921 km/s) Earth entry.

quadratic equation in $\sin\theta$ with the solution for positive $\sin\theta$ as follows:

$$\sin\theta = \frac{-\tfrac{1}{2}L/D\cos\theta_E + \sqrt{(\tfrac{1}{2}L/D)^2 + \sin^2\theta_E}}{1 + (\tfrac{1}{2}L/D)^2}$$

However, the method of analysis utilizing the numerical method has the advantage that it can be readily generalized to solutions for peak heating, for example.

In Fig. 2, for some selected values of L/D, the solution of Eq. (9′) is displayed for given values of θ_E. The exact solution is compared with the approximate analytic solution of Ref. 1 for $L/D = 0.25$ in Fig. 3, and for the approximate analytic solution of Ref. 2 for $\theta_E = 12$ deg in Fig. 4. As pointed out previously, the solutions of Ref. 1 are valid for θ_E near zero and those of Ref. 2 are valid for L/D near zero. The figures illustrate this clearly. The solutions of Eq. (9′) were also employed to evaluate the peak acceleration for $L/D = 0.25$ and 1.0 utilizing Eq. (12). In Fig. 5, the values of peak acceleration obtained were compared with the solutions given in Ref. 1 for $V_E = 26{,}000$ fps (7.92 km/s) for Earth entry.

Concluding Remarks

This paper presents a unified treatment of the effect of lift on peak acceleration during atmospheric entry by employing a general expression for the peak acceleration. The transcendental equation which arises in the analysis is solved without invoking any approximations. The connection with earlier studies is pointed out and results are presented and compared with earlier studies. The method of analysis can easily be generalized to obtained solutions for peak heating, for example.

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